Costly Entry into Oligopolistic Markets*

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[This draft is work in progress and is very incomplete]

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Abstract

We study firms entry decisions into oligopolistic markets under private information. Firms are heterogeneous, differing in their distribution of private information, entry costs and reaction to competition. Post-entry profits depend on the firm's identity, and the identities and private information of the other firms in the market. We introduce a notion of player's *strength*, and show that an equilibrium where players' strategies are ranked by strength, or *herculean equilibrium*, exists. Even though all equilibria are *ex-post* inefficient, there is an *ex-ante* efficient equilibrium. When profits are not too elastic with respect to the firm's private information, the herculean equilibrium is the unique equilibrium of the game and, consequently, efficient.

Keywords: Costly entry, oligopolistic markets, private information, efficiency

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1 Introduction

This paper studies firms entry decisions into oligopolistic markets. We depart from traditional entry models, such as Bresnahan and Reiss (1990, 1991) and Berry (1992), in two ways. First, we allow firms to be privately informed about how valuable is for them to participate in the market before making their entry decisions. In addition, we allow firms to be heterogenous in several ways: firms may have different entry costs, different distributions of private information, and different responses to entry and private information of the opponents. These generalizations enable to study entry into a wider class of markets, expanding the scope of empirical applications.

We show that every entry game has an equilibrium and and that every equilibrium is in cutoff strategies, i.e., players enter with certainty if their private information is above some given threshold. To further characterize firms entry decisions, we develop a notion of *strength*, which uses the firms' profit functions, distributions of valuations and entry costs to rank the firms. We show an equilibrium in which stronger firms are more like to enter, or *herculean equilibrium*, always exists. Furthermore, we show that when the profit function is not too elastic with respect to the firm's private information, the herculean equilibrium is the unique equilibrium of the game. We also investigate the efficiency properties of the entry decisions. We show that, although every equilibrium is *ex-post* inefficient, the entry game always has an *ex-ante* efficient equilibrium. Therefore, when the conditions to have a unique equilibrium hold, the herculean equilibrium is the unique equilibrium of the game and, thus, is *ex-ante* efficient.

From a theoretical standpoint, our results allow to incorporate richer models of entry into competition policy analysis, market design, or applied theory in general. The introduction of heterogeneity into the single-market entry model is a fundamental step to develop a dynamic theory of market entry. For instance, heterogeneity naturally emerges when firms face a sequence of non-independent markets. Take, for instance, the airline industry. Even if firms are symmetric before entering any market, the network formed by their current routes conditions the profitability of entering into a new route, making firms heterogeneous before making their entry decisions.

For empirical analysis, our results provide methodological suggestions on how to examine entry. First, our results suggest that herculean equilibria should be focal at an empirical analysis of market entry. This is so, as the existence of an herculean equilibrium is guaranteed and, moreover, it maybe the only equilibrium of the game. More importantly, our conditions for uniqueness provide a foundation for robust counterfactual analysis. In particular, when conditions for uniqueness hold, our results guarantee that the predictions are well defined as they corresponds to the unique prediction of the model.

The rest of this paper proceeds as follows. Section 2 introduces the model, Section 3 shows equilibrium existence and that every equilibrium is in cutoff strategies. Section 3 also introduces our notion of strength. To exemplify our findings, and because is an important application, Section 4 studies herculean equilibrium in the context of a second price auction. Section 5 extends the results to general markets and Section 6 concludes.

1.1 Literature Review

In the context of firms entry decisions, structural techniques began with Bresnahan and Reiss (1990) who studied a statistical model of monopolistic entry and Bresnahan and Reiss (1991) and Berry (1992) which extended the monopoly entry model to a complete information scenario in which two or more symmetric firms may enter the market. One drawback of the complete information model is that possess multiple equilibria. Multiplicity of equilibria weakens the empirical identification of the model, limiting what can be empirically learned from it. Mazzeo (2002) studies the entry problem and product choice under the assumption that firms always choose to behave according to one of the multiple equilibria of the entry game. Tamer (2003) shows that, without further assumptions, multiple equilibria leads to set identification of parameters instead of point identification. Ciliberto and Tamer (2009) apply the previous methodology to the airline industry. Some recent work has study entry in the context of private information. Seim (2006) studies entry decisions and product choice. Athey *et al.* (2011) and Krasnokutskaya and Seim (2011) study the case of auctions.

An important application of our results are second price auctions with participation costs. In this context, Samuelson (1985) studies the symmetric case, i.e., all bidder have the same participation costs and the same distribution of valuations. Tan and Yilankaya (2006) and Cao and Tian (2013) study uniqueness of equilibrium under different forms of bidder heterogeneity. The former paper restricts attention to auctions where players have identical participation costs but the distribution of valuations of one bidder first order stochastically dominates the other bidder. The latter paper studies an environment where all agents draw their valuations from the same distribution but differ in their participation costs. In this respect, our contribution is to allow consumers to have any arbitrary pair of distribution of valuations and participation costs. McAfee and McMillan (1987) and Tan (1992) study auctions with participation costs where agents take the participation decision before observing their own valuation.

Our welfare analysis expands the early work of Stegeman (1996) who studies efficiency in first and second price auctions with participation costs. In the context of a second price auction, he gives conditions to have an *ex-ante* efficient equilibrium. He also relates the existence of an *ex-ante* efficient equilibrium in the first price auction to the existence of a symmetric efficient allocation in the second price auction. Our results extend his by proving the unconditional existence of an *ex-ante* efficient equilibrium in a wider class of entry games. Also, by finding conditions to have a unique equilibrium, we partially identify the efficient equilibrium.

2 A Model of Market Entry

Consider n firms simultaneously deciding on whether to enter a market. Each firm possesses private information about how much it values to participate in the market. Entry is costly and the entry cost $c_i > 0$ may differ among firms. The post-entry profits of firm i depend on every firm entry decision, i's private information $v_i \sim F_i$ and, the private information of the participating firms. We assume that the distribution of private information F_i is an atomless and continuously differentiable distribution function with full support on $\mathbb{R}_+$.[1]

Let $e_i \in \{0, 1\}$ be an indicator function that takes the value of one if firm i enters the market. Denote by $e = (e_1, e_2, \ldots, e_n)$ the vector of (ex-post) entry decisions, and let $E_i = \{e : e_i = 1\}$ be the set of all possible market structures in which firm i participates. For a given market structure e, define $I_i(e) = \{j \neq i : e_j = 1\}$ and $O_i(e) = \{j \neq i : e_j = 0\}$ to be the set of i's competitors that (respectively) are in or out of the market. Finally, define $v = (v_1, v_2, \ldots, v_n)$ to be the vector of private information, and v_{-i} to be the private information of all firms but i.

Firm i's post-entry profits of participating in the market when the realized

[1] All our results would go through if the support of F_i were the interval $[0, b_i]$ with $b_i > 0$. This formulation, however, complicates exposition as we would have to explicitly deal with corner solutions.

vector of private information is v and firm's entry decisions is represented by e, is given by $\pi_i(v, e)$. We assume that the profit function is integrable, has finite expectations and that satisfies the following properties:

A1 Conditional on i's entry (i.e., $e_i = 1$), $\pi_i(v, e)$ is a strictly-increasing differentiable function of v_i. If a firm stays out, it receives zero profit.

A2 Conditional on i's entry: for each $j \in O(e)$, $\pi_i(v, e)$ is constant in v_j, and; for all j, $\pi_i(v, e)$ is weakly decreasing in e_j.

A3 There exists v_i such that $\lim_{v_{-i}\to\infty} \pi_i(v_i, v_{-i}, \mathbf{1}) > c_i$, where $\mathbf{1}$ is a n-dimensional vector of ones. For every v_{-i} and e, $\pi_i(0, v_{-i}, e) < c_i$.

Assumption one states that, upon participating in the market, firms' profits are increasing in their private information, regardless of the realized market structure. The second assumption says that profits decrease when a new competitor enters the market, and that the private information of non-participating firms is payoff-irrelevant. The third assumption guarantees that, regardless of the market structure, every firm may enter if its private information is sufficiently high, and that the entry cost is meaningful. Observe that we impose no restriction on how the profits firm i react to the private information of a participating firm v_j. This formulation allows for substitution and certain degree of complementary in the private information of the opponents.

Before making any entry decision, each firm observes its private information v_i. Upon observing v_i, each firm independently and simultaneously decides whether to enter the market. If firm i enters the market, it incurs on the entry cost c_i. The tuple $(\pi_i, F_i, c_i)_{i=1}^n$ is commonly known by all the firms in the market.

An entry strategy for firm i is a mapping from the firm's private information v_i to a probability of entering in the market $p_i : \mathbb{R}_+ \to [0, 1]$. We assume that the strategy of player i is an integrable function with respect to her own type v_i. We study the Perfect Bayesian Equilibrium of the entry game.

Define $G_i(v_{-j})$ to be joint distribution of the private information of i's opponents. Denote by $p = (p_1, p_2, \dots, p_n)$ the vector of strategies. The expected profits of firm i *after* drawing the private information v_i but *before* the entry decisions are realized is:

$$\Pi_i(v_i, p) = p_i(v_i) \left[\sum_{e\in E_i} \int_0^\infty \pi_i\left(v_i, v_{-i}, e\right) \Pr[e|p, v_{-i}] dG_i(v_{-i}) - c_i \right].$$

where $\Pr[e|p, v_{-i}]$ is the probability of observing market structure e, given the vector of strategies p and the realization of private information v_{-i}. In words, conditional on entry, the expected profits of firm i consists on the expected sum of profits that firm i would get under each market structure, integrated over all possible realizations of the opponents private information, minus cost of entry c_i.

This formulation accommodates several sources of heterogeneity among firms. Firms can differ in their distribution of private information which represent that firms may have access to different production technologies or variety of products. Firms can be heterogeneous in their entry costs representing, for instance, that firms may have already incurred in complementary investments or have incurred in sunk costs. Finally, firms may differ on how they react to competition. For instance, our model accommodates for product differentiation or that firms may take a turns on who plays first in the post entry market (i.e., competition à la Stackelberg). Since post entry profits are abstract and general they contains several forms of product market competition that used in applied work, for example:

Example 1 (Homogeneous-good Cournot competition)**.** Let $P(Q)$ be the inverse demand function satisfying standard uniqueness and stability assumptions.[2] The cost structure $C_i(q_i, v_i)$ with both total and marginal cost decreasing in v_i and $C_i(q_i, 0)$ being prohibitively high. The induced profit function satisfies all the assumptions of the model for sufficiently small c_i.

Example 2 (Homogeneous-good Bertrand competition)**.** Let $D(P)$ be a strictly decreasing demand function satisfying $\lim_{P\to\infty} D(P) = 0$. Consumers choose the firm with the lowest price, or split evenly in case of a tie. The marginal cost of firm i is equal to $1/v_i$. Then, for sufficiently small entry cost, the equilibrium profit function satisfies all assumptions of the model.

Example 3 (Second-price auction)**.** There is a product to be auction. Each firm values the product in $v_i \sim F_i$. Assuming that after entry each firm bids its weakly dominant strategy, i.e., its valuation. The post-entry profit function is given by $\pi_i(v, e) = \max\{0, e_i v_i - \max\{e_j v_j\}_{j\neq i}\}$ which satisfies all assumption for any $c_i > 0$.

[2]For all Q such that $P(Q) > 0$: i) $P'(Q) + qP''(Q) < 0$ for all $q \in [0, Q]$. ii) $\lim_{Q\to\infty} P(Q) = 0$.

3 Peliminary Results

In this section we provide a general characterization of all equilibria in the game. We prove that, without loss of generality, we can restrict attention to cutoff strategies. In addition, we prove that an equilibrium always exist.

Definition. A strategy $p_i(v_i)$ is called *cutoff* if there exists $x > 0$ such that

$$p_i(v_i) = \begin{cases} 1 & \text{if } v_i \geq x \\ 0 & \text{if } v_i < x \end{cases}.$$

A cutoff strategy specifies whether a firm enters a market with certainty depending on its private information being above some given threshold. The next Proposition shows that, without loss of generality, we can restrict our attention to study cutoff strategies, and that an equilibrium always exists.

Proposition 1. *For any game* $(\pi_i, F_i, c_i)_{i=1}^n$ *there exists an equilibrium. Every equilibrium of the game is in cutoff strategies.*

In any best response, when a firm's private information is equal to her cutoff strategy, the firm is indifferent to enter the market or not. We break this indifference by assuming that firms always participate at their cutoff.

From now and on, we will abuse notation and denote a cutoff strategy by the cutoff itself. In addition, and without loss of generality, we order the firms identities according to their cutoffs, with x_1 being the firm with the smallest cutoff and x_n the firm with the largest. Let $n(e)$ represent the number of firms entering the market under structure e, and let $G_{I(e)}(\mathbf{v}_{n(e)})$ where $\mathbf{v}_{n(e)}$ is a $n(e)$-dimensional vector be the joint distribution of private information of the firm that participate in the market under structure e. The next Lemma characterizes all cutoff equilibria.

Lemma 1. *Let* $x_1 \leq x_2 \leq \cdots \leq x_n$ *be cutoff strategies. They constitute an equilibrium if and only if the following condition holds for each player* i*:*[3]

$$\sum_{e \in E_i} \left(\prod_{j \in O(e)} F_j(x_j) \right) \int_{\{x_j\}_{j \in I(e)}}^{\infty} \pi_i \left(x_i, \mathbf{v}_{n(e)}, e \right) d^{n(e)} G_{I(e)}(\mathbf{v}_{n(e)}) = c_i. \tag{1}$$

where the integral is across the $n(e)$ *dimensions.*

[3] The following notation conventions are used throughout the paper: $\sum \emptyset = 0$ and $\prod \emptyset = 1$.

Lemma 1 characterizes all equilibria of the game. It does not, however, provide any information about which firms play which cutoff, and how many equilibria in the entry game exists. The next definition ranks players according to the game fundamentals, and will help us to further characterize market entry.

Definition. For a given game $(\pi_i, F_i, c_i)_{i=1}^n$, the *strength* of player i is the unique number $s_i \in \mathbb{R}_+$ that solves

$$\sum_{e \in E_i} \left(\prod_{j \in O(e)} F_j(s_i) \right) \int_{s_i}^{\infty} \pi_i \left(s_i, \mathbf{v}_{n(e)}, e \right) d^{n(e)} G_{I(e)}(\mathbf{v}_{n(e)}) = c_i. \tag{2}$$

We say that player i is *stronger* than player j if $s_i \leq s_j$. An equilibrium in cutoffs strategies is called *herculean* if the cutoffs defining the equilibrium strategies are ordered by strength, with stronger players playing lower cutoffs.[4]

Conditional on the opponents' strategies, a cutoff strategy is determined by the value of the private information that makes a firm indifferent between participating in the market or not. The notion of strength ranks firms in terms of the cutoffs they would play under the assumption that every firm in the market is playing the same cutoff strategy. Intuitively, strength ranks firms according to their ability to endure competition. Stronger firms need lower draws to enter the market and, in context where the distribution of private information are symmetric ($F_i = F$ for all i), stronger firms are more likely to participate than their weaker opponents.

If equilibrium cutoffs are ordered according to the firms' strength, we call it herculean. We will show that herculean cutoffs emerge naturally in entry games, and that, under certain conditions, is the unique equilibrium of the entry game. The next corollary help to develop some intuition of the notion of strength and herculean equilibrium.

Corollary 1. *If every firm is equally strong (i.e., $s_i = s$ for all i), then $x_i = s$ is an (herculean) equilibrium of the entry game.*

The corollary is a straight consequence of the definition of strength and Lemma 1. A particular case in which every firm is equally strong is a symmetric game. Thus, in symmetric games, the notions of strength, herculean equilibrium and symmetric

[4]Strength is always well defined due to the intermediate value theorem. The left hand side of (2) is continuous and, by A3, less than c_i when $s_i = 0$ and larger than c_i for some sufficiently high s_i.

equilibrium coincide. Our notion of strength of firm i is precisely the answer to the question, what would the symmetric equilibrium be if the game is symmetric and given by the fundamentals of firm i. Thus, strength and herculean equilibrium, build an asymmetric analog of the symmetric equilibrium in an asymmetric game.

4 2^{nd}-Price Auction with Entry Costs

We start by examining a 2^{nd}-price auction with entry costs. Our motivation to do this is three-fold. Due to its simplicity, it will help to better illustrate our results. Also, 2^{nd}-price auctions are very common and an important application in practice. Finally, in this scenario we are able to obtain sharper results.

In the case of a second price auction, the notion of strength of player i becomes the unique number $s_i \in \mathbb{R}_+$ that solves

$$s_i \prod_{k \neq i} F_k(s_i) = c_i.$$

Define $A_i^n = \prod_{k>i}^n F_k(x_k)$, which corresponds to the probability that bidders playing cutoffs above x_i do not participate in the auction. Define $B_i(v) = \prod_{\ell<i} F_\ell(v)$, which is the probability that players playing cutoffs below x_i obtain a valuation lower than v. Finally, define

$$D_i = x_i B_i(x_i) - \sum_{k=1}^{i-1} \left(\prod_{\ell>k}^{i-1} F_\ell(x_\ell) \right) \int_{x_k}^{x_{k+1}} s dB_{k+1}(s)$$

which represents the weakest player's expected payoff in an auction with $n = i$ bidders when its valuation is equal to its cutoff. The next lemma characterizes all cutoff equilibria in second price auctions.

Lemma 2. *Let $x_1 \leq x_2 \leq \cdots \leq x_n$ be cutoff strategies. In a second price auction they satisfy*

$$A_i^n D_i = c_i \tag{3}$$

4.1 Auctions with two players

To better illustrate our results we start with a two-player auction. The following proposition is a version of our main result.

Proposition 2. *There always exists an herculean equilibrium. Each herculean equilibria is characterized by cutoffs $x_1 \leq x_2$ that jointly solve*

$$\begin{aligned} x_1 F_2(x_2) &= c_1 \\ x_2 F_1(x_2) - \int_{x_1}^{x_2} v dF_1(v) &= c_2. \end{aligned} \tag{4}$$

Moreover, if F_1 and F_2 are concave, an herculean equilibrium is the unique equilibrium of the game.

There are two messages in Proposition 2. First it tells us that herculean equilibria are focal in the sense that their existence are guaranteed. It also, shows the power of the strength notion, it reduces a combinatorial problem—$n!$ different orders for the n firms cutoffs—to a problem of order n. In addition, the proposition tells us that concavity of the distribution functions is sufficient to guarantee uniqueness of equilibrium.

Proposition 2 is a generalization of Tan and Yilankaya (2006) and Cao and Tian (2013). Both papers develop notions of strength in particular environments. Tan and Yilankaya (2006) assume that each bidder has the same participation costs (i.e., $c_i = c$ for all i), and call bidder i stronger than j if F_i First Order Stochastically Dominates (FOSD) F_j. Under the symmetric cost assumption, FOSD implies $s_i \leq s_j$; however, the converse is not true. Similarly, Cao and Tian (2013) study an environment in which all valuations distribute symmetrically (i.e., $F_i = F$ for all i) and define bidder i to be stronger than j if $c_i < c_j$. Our definition encompasses both definitions, generalizing it to deliver a complete ranking for any game $(F_i, c_i)_{i=1}^2$.

Herculean equilibrium also relates to efficiency. As Stegeman (1996) pointed out, when entry is costly, the equivalence between *ex-ante* and *ex-post* efficiency is broken. To illustrate this point Figure 1.a depicts an equilibrium with two players ($n = 2$), equal participation costs ($c_i = c$), but different cutoff equilibrium strategies ($x_1 < x_2$). Three types of inefficiencies arise: (i) the dotted area represents situations in which player one participates and has a lower valuation than player two, which does not participate, so the object is assigned to the player with the lowest valuation; (ii) represents a situation in which both players participate, paying excessive participation costs (lightly-shaded area); (iii) represent realizations in which there is no participation although it is efficient that one player participates and obtains the good (dark-shaded area).

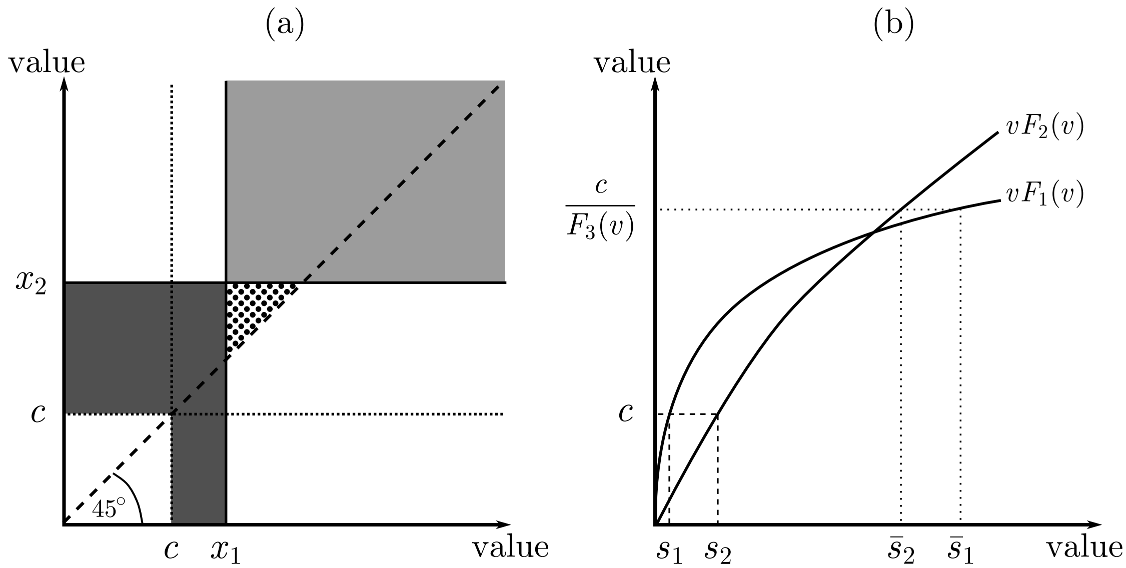

Figure 1: (a) *ex-post* inefficiency. (b) *Strength* is a local condition.

Since players have to make participation decisions without knowing whether other players are participating or not, there will always exits realization for the profile of valuation that will lead to excessive entry when players have high valuations (area (ii)), and to insufficient entry when all players have low valuations (area (iii)). On the other hand, inefficiencies of they type (i) will only emerge in equilibria in which players play different cutoffs.

From an *ex-ante* perspective, however, there is an efficient equilibrium. Consider the problem that a planner faces when she can choose a strategy for each player but only conditioning on the players private information, i.e., the planner chooses a set of functions $p_i : \mathbb{R}_+ \rightarrow [0,1]$ determining the probability that player i participates given her valuation. Using similar arguments to those in Proposition 1 can be shown that the planner will only consider cutoff strategies. Therefore, the planner chooses the vector of cutoffs $x = (x_1, x_2, \ldots, x_n)$ that maximizes

$$W(x) = \sum_{i=1}^{n} \left[\int_{x_i}^{\infty} (v_i H_i(v_i) - c_i)\, dF_i(v_i) \right] \tag{5}$$

where $H_i(v_i) = \prod_{k \neq i} F_k(\max\{v_i, x_k\})$ is the probability that player i obtains the object when her valuation is v_i. To understand the previous expression start by observing that transfers between players are not relevant in terms of welfare. Lets focus on the planners' payoffs coming from player i. With probability $dF_i(v_i)$, player i draws the valuation v_i and participates in the auction whenever $v_i \geq x_i$, in which case she pays the costs of participating c_i and wins the object with probability $H_i(v_i)$.

Proposition 3. *There exists an equilibrium that is* ex-ante *efficient. Every equilibrium of the game corresponds to either a (possibly local) maximum, minimum, or saddle point of the social welfare function.*

The entry game has always an ex-ante efficient equilibrium. Thus, when there is a unique equilibrium on the game, it must be efficient. Since an herculean equilibrium always exists, when uniqueness holds the equilibrium is both herculean and efficient. The proposition, however, does not tell us that an herculean equilibrium is always ex-ante efficient. It is not hard to build an example of a symmetric game where the symmetric equilibrium is inefficient. Thus, the herculean equilibrium is not always efficient.

4.2 Three or more Players

This section studies equilibria in the case of $n > 2$ players. Unfortunately, the notion of strength is local and cannot always rank players 1 and 2 independently of player 3's distribution function. To see this, consider picture Figure 1.b. There we depict the strength of player 1 and 2 (s_1 and s_2 respectively) under symmetric costs in two and three player case. With a third player the strength of player $i \in \{1, 2\}$ is similar the two player case but scaling the cost by $c/F_i(s_i)$. In the picture we can see that even with concave distribution functions we can have a reversal of strength.

There are, however, interesting situations relevant for applications in which strength is sufficient to rank players and characterize equilibrium. Section 4.2.1 characterizes the equilibrium restricting attention to an scenario in which all players belong to one of two groups (F_1, c_1) and (F_2, c_2). Section 4.3 characterizes equilibrium under conditions that guarantee the robustness of the ranking given the strength of the players.

4.2.1 Two Groups of Players

Suppose there exist two groups of players characterized by pairs (F_1, c_1) and (F_2, c_2), with $s_1 \le s_2$, i.e., players in group 1 are stronger than those in group 2. Let m_i be the number of players belonging to group i. The following proposition characterizes all herculean equilibria in this case.

Proposition 4. *There always exists an herculean equilibrium, which are given by the cutoffs $x_1 \leq x_2$ that jointly solve*

$$x_1 F_1(x_1)^{m_1-1} F_2(x_2)^{m_2} = c_1 \tag{6}$$

$$F_2(x_2)^{m_2-1}\left[x_2 F_1(x_2)^{m_1} - \int_{x_1}^{x_2} v d\left(F_1(v)^{m_1}\right)\right] = c_2. \tag{7}$$

Moreover, if F_1 and F_2 are concave the herculean equilibrium is the unique equilibrium of the game.

4.3 Robust Order Among Players

The notion of strength order players based on a local condition. The following condition guarantee the robustness of strength globally. For all $v \geq 0$

$$F_1(v)c_1 \leq F_2(v)c_2 \leq \cdots \leq F_n(v)c_n. \tag{8}$$

Observe that previous condition is a n-player generalization of Tan and Yilankaya (2006) and Cao and Tian (2013).

Proposition 5. *Suppose that condition (8) holds. Then, firms are ordered by strength with bidder one being the strongest player. An herculean equilibrium exits and, if F_i are concave, there is a unique herculean equilibrium.*

5 Herculean Equilibrium in the General Model

5.1 Two Potential Firms

In this section we start by generalizing our two potential firm to general product market competition. In this case we greatly simplify notation. Let $\pi_i(v_i)$ be the profits of firm i when it is a monopolist in the market and $\pi_i(v_i, v_j)$ be the duopolistic profits when facing j as a competitor. In this scenario the strength of firm i is the unique number $s_i \in \mathbb{R}_+$ that solves:

$$\pi_i(s_i)F_j(s_i) + \int_{s_i}^{\infty} \pi_i(s_i, y) dF_j(y) = c_i$$

Let $\Delta_i(x, y) = \pi_i(x) - \pi_i(x, y)$ be the incremental rent of going from a duopoly to a monopoly when facing a competitor with private information y.

Proposition 6. *There always exists an herculean equilibrium. Each herculean equilibria is characterized by the cutoffs $x_1 \leq x_2$ that jointly solve*

$$\pi_i(x_i)F_j(x_j) + \int_{x_j}^{\infty} \pi_i(x_i, y)\, dF_j(y) = c_i. \tag{9}$$

Moreover, the herculean equilibrium is the unique equilibrium in the game if for all y and x

$$\frac{f_i(x)}{F_i(x)} \frac{\Delta_i(x, y)}{\pi_i'(x)} < 1. \tag{10}$$

Proposition 6 extends the previous result to general market structures. Sufficient condition (10), however, is hard to interpret. Next result allow us to give clarity on the matter.

Proposition 7. *The herculean equilibrium is the unique equilibrium in the game if any of the following conditions hold for all x and y:*

$$a)\ \varepsilon_{F_i,\pi_i} = \frac{f_i(x)}{F_i(x)} \frac{\pi_i(x)}{\pi_i'(x)} < 1$$
$$b)\ \varepsilon_{F_i,\Delta_i} = \frac{f_i(x)}{F_i(x)} \frac{\Delta_i(x, y)}{\Delta_i'(x, y)} < 1$$

Condition a) tells us that when the monopolistic profits have decreasing returns to the private information, we have a unique entry equilibrium. In the context of a first price auction, the monopolistic function is linear, so that $\pi_i(x)/\pi_i'(x) = x$ and the condition translate to concavity of the distribution function. Similarly, condition b tells us that is also sufficient to have decreasing returns to the private information in the context of a duopoly.

A subtle, but important question is whether firms can be ranked in terms of their expected ex-ante profits, i.e., the expectation of the profit before the realization of the private information. It turns out, that a clear ranking only exists in symmetric games. In particular, when considering asymmetric equilibria, the expected profits of the players can be inversely ranked according to their equilibrium cutoff—i.e., players with lower cutoffs have greater expected profits. However, this is not true with asymmetric players.

Proposition 8. *In a symmetric game, players ex-ante profits can be ranked in terms of their cutoffs: players playing the lowest cut-off obtain higher expected profits. In an asymmetric game, cutoff ranking does not relate with profit ranking.*

An example that asymmetric players profit are not ranked can be easily constructed. For instance, consider a two player second price auction, and let $F_1(v) = 1 - e^{-v/2.8854}$, $F_2(v) = v/(v+2)$ and $c_1 = c_2 = 1$. Then, both players have equal strength ($s_1 = s_2 = 2$), and $x_1 = x_2 = 2$ is the unique equilibrium of the game since both distribution functions are concave. However, ex-ante profits are not the same as in the range $v \geq 2$, F_2 first order stochastically dominates F_1 implying that player's two ex-ante profit is higher.

5.1.1 Two Groups of Players

Suppose there exist two groups of firms characterized by the pairs (π_1, F_1, c_1) and (π_2, F_2, c_2), with $s_1 \leq s_2$, i.e., players in group 1 are stronger than those in group 2. Let m_i be the number of firms belonging to group i. In addition assume that π_i is symmetric to the private information of competitors within a class. The following proposition characterizes all herculean equilibria in this case.

Proposition 9. *An herculean equilibrium always exists and is given by the cutoffs* $\mathbf{x} = (x_1, x_2)$ *that jointly solves for each firm* i*:*

$$\sum_{k=0}^{m_j} \left\{ \binom{m_j}{k} F_j(x_j)^{m_j-k} \left[\sum_{r=0}^{m_i-1} \binom{m_i-1}{r} F_i(x_i)^{m_i-1-r} \mathbb{E}[\pi_i(x_i)|\mathbf{x}, r, k] \right] \right\} = c_i. \tag{11}$$

where

$$\mathbb{E}[\pi_i(v)|\mathbf{x}, r, k] = \int_{x_1}^{\infty} \int_{x_2}^{\infty} \pi_i(v, \mathbf{y}_r, \mathbf{z}_k) d^k F_j(z) d^r F_i(y).$$

Moreover, the herculean equilibrium is the unique within-group symmetric equilibrium in the game if for all y *and* x

$$\frac{f_i(x)}{F_i(x)} \frac{\pi_i(x) - \pi_i(x,y)}{\pi_i'(x)} < \frac{1}{m_j}. \tag{12}$$

Appendix

Preliminary Results

Lemma 3. *The following properties hold.*

1. $x_i B_i(x_i) \geq D_i$ *and strict if exists* $j < i$ *such that* $x_j < x_{j+1}$.

2. *If* $x_i > x_{i_1}$, $D_i > F_{i-1}(x_{i-1})D_{i-1}$ *and with equality if* $x_i = x_{i_1}$.

Lemma 4. *Let* $H_i(x_i, x_j)$ *be defined by (13). Then the partial derivatives with respect to* x_i *and* x_j *are postive and equal to:*

$$\frac{\partial H_i}{\partial x_j} = \sum_{k=0}^{m_j} \left\{ \binom{m_j}{k} (m_j - k) F_j(x_j)^{m_j-k-1} f_j(x_j) \mathbb{E}_r \left[\mathbb{E}[\Delta_i(x_i, x_j)|\mathbf{x}, r, k] \right] \right\}$$

$$\frac{\partial H_i}{\partial x_i} = \sum_{k=0}^{m_j} \left\{ \binom{m_j}{k} F_j(x_j)^{m_j-k} \left[\sum_{r=0}^{m_i-1} \binom{m_i-1}{r} F_i(x_i)^{m_i-2-r} \left(F_i(x_i)\mathbb{E}[\pi_i'(x_i)|\mathbf{x}, r, k] \right. \right. \right.$$
$$\left. \left. \left. + \ (m_i - 1 - r) f_i(x_i) \mathbb{E}[\Delta_i(x_i, x_i)|\mathbf{x}, r, k] \right) \right] \right\}$$

where

$$\mathbb{E}_r \left[\mathbb{E}[\pi_i(x_i)|\mathbf{x}, r, k] \right] = \sum_{r=0}^{m_i-1} \binom{m_i-1}{r} F_i(x_i)^{m_i-1-r} \mathbb{E}[\pi_i(x_i)|\mathbf{x}, r, k]$$

Lemma 5. *Condition (12) implies*

$$(m_j - k) f_i(x_i) \mathbb{E}[\Delta_i(x_i, x_j)|\mathbf{x}, r, k] < F_i(x_i) \mathbb{E}[\pi_i'(x_i)|\mathbf{x}, r, k]$$

Proof. Use Leibniz differentiation and the following property of binomial coefficients □

Omitted Proofs

Proof of Proposition 1 Fix any p_{-i}. We show that i's best response is a cutoff strategy. Because i's profit is linear in p_i, i's best response is to participate with probability one whenever there is a positive payoff of doing so. Conditional on firm i entering with certainty ($p_i(v) = 1$), $d\Pi(v_i, p)/dv_i > 0$ and i's profits are strictly increasing in its type. A3 implies that $\Pi(0, p_i = 1, p_{-i}) < 0$, and that exits v sufficiently high such that $\Pi_i(v, p) > c_i$. Thus, $\Pi(v_i, p_i = 1, p_{-i})$ single crosses zero and i's best response is a cutoff strategy which is given by the valuation x that satisfies $\Pi(x, p_i = 1, p_{-i}) = 0$.

For existence, we check the conditions for Brouwer's fixed-point Theorem to apply. Because F_i is atomless and has full support, player i's best response is continuous in each of the opponent cutoffs. A2 implies that firm's i lowest profit is attained when each opponent enters the market with certainty (i.e., $p = \mathbf{1}$ for all v). Let K_i be lowest type of firm i that satisfies $\Pi_i(K_i, \mathbf{1}) = 0$ which exists by A3. The vector of best responses is a continuous mapping from the compact and convex set $\times_{i=1}^n [0, K_i]$ to itself and the conditions for the Theorem are met.

Proof of Lemma 1 By Proposition 1 cutoff strategies satisfy $\Pi(x, p_i = 1, p_{-i}) = 0$. By A2, firms that stay out of the market are payoff irrelevant. By the cutoff structure $\Pr[e|p, v_i]$ is either zero or one. Integrating over payoff irrelevant firms, delivers (1).

Proof of Proposition 2

Existence. If $s_1 = s_2$ the result is straight forward. Assume $s_1 < s_2$, let $g(x)$ the function implicitly defined by

$$g(x)F_2(x) = c_1.$$

The function $g(x)$ is strictly decreasing in x and satisfies $g(s_1) = s_1$. Define the function $h : [s_1, \infty) \to \mathbb{R}$ by

$$h(x) = \left[xF_1(x) - \int_{g(x)}^{x} ydF_1(y) \right] - c_2.$$

The function $h(x)$ is continuous and corresponds to the utility of firm 2 playing the cutoff strategy $x_2 = x$ when firm 1 best respond to x (i.e., $x_1 = g(x)$). The next two claims prove the result.

Claim 1. $x_2 \in [s_1, \infty)$ is necessary and sufficient to have herculean cutoffs.

Proof. $g(x)$ is weakly decreasing in x and $g(s_1) = s_1$. Therefore, $x_1 < x_2$ iff $x \geq g(x)$ iff $x_2 \in [s_1, \infty)$. □

Claim 2. $h(s_1) < 0$ and $h(x)$ is unbounded above.

Proof. Firm 2 being the weak implies $h(s_1) = s_1F_1(s_1) - c_2 < 0$. On the other hand, $h(x)$ is unbounded above as $xF_1(x)$ is unbounded and A3. □

By the intermediate value theorem, Claim 6 plus continuity imply that there exists $x^* \in (s_1, \infty)$ such that $h(x^*) = 0$. On the other hand, $h(x^*) = 0$ holds iff equations (4) hold. Therefore, by Claim 5, we have an herculean equilibrium with $x_1 = g(x^*)$ and $x_2 = x^*$. ■

Uniqueness. The uniqueness proof is divided in two claims. Concavity is used in each of them.

Claim 3. There exists a unique herculean equilibrium.

Proof. In order to have a unique equilibrium that is herculean, it is sufficient to show that $h'(x) > 0$ for all $x \geq s_1$, so that $h(x)$ single crosses zero

$$h'(x) = F_1(x) + g'(x)g(x)f_1(g(x)).$$

Implicitly differentiating $g(x)$

$$g'(x) = -g(x)f_2(x)/F_2(x)$$

replacing back in to the expression above delivers

$$h'(x) = F_1(x) - \frac{f_2(x)g(x)^2 f_1(g(x))}{F_2(x)}. \tag{13}$$

It is shown that a lower bound for the expression above is always positive. Using concavity ($xf_i(x) \leq F_i(x)$), maximize the subtracting by substituting $g(v)f_1(g(v))$ for $F_1(g(v))$ in the denominator and $f_2(x)/F_2(x) \leq x^{-1}$. Then, equation (15) becomes

$$h'(x) \geq F_1(x)\left(1 - \frac{g(x)}{x}\right).$$

Since $x \geq g(x)$ for $x \geq s_1$, the lower bound is positive and $h'(x) > 0$. □

Claim 4. There is no equilibrium in which the strong firm plays a higher cutoff than the weak firm.

Proof. To prove that the only equilibrium is the herculean, suppose we have a non-herculean equilibrium, i.e., $x_1 > x_2$ but $s_1 < s_2$. Define $\bar{g}(x)$ to be the function that satisfies

$$\bar{g}(x)F_1(x) = c_2.$$

Similarly, define

$$\bar{h}(x) = xF_2(x) - \int_{\bar{g}(x)}^{x} y dF_2(y) - c_1.$$

which corresponds to the utility of the strong player, playing the cutoff x when the weak player plays best responds to x by playing $\bar{g}(x)$. Following analogous steps to those in Claim 5 it can be shown that in order to have a non-herculean equilibrium, $\bar{h}$ has to be defined on $[s_2, \infty)$, i.e., any $x < s_2$ would deliver an herculean cutoff as a candidate and this is ruled out by assumption. Now observe that $s_1 < s_2$ implies that $\bar{h}(s_2) = s_2F_2(s_2) - c_1 > 0$. By an analogous argument given in Claim 7, concavity implies $\bar{h}'(x) > 0$, and $\bar{h}(x) > 0$ for all $x \in (s_2, \infty)$, so there is no x^* such that $\bar{h}(x^*) = 0$ and no non-herculean equilibrium exists. □

Proof of Proposition 3 Without loss of generality, order the players identities from the lowest cutoff chosen by the planer x_1, to the highest x_n. Differentiating

(5) with respect to x_i we obtain

$$W_{x_i} = f_i(x_i)(c_i - x_i H_i(x_i)) + \sum_{k \neq i} \int_{x_k}^{\infty} v_i \left(\frac{dH_k(v_i)}{dx_i} \right) dF_k(v_i).$$

Observing that $dH_k(v_i)/dx_i = f_i(x_i) \prod_{\ell \neq k,i} F_\ell(\max\{v_i, x_\ell\})$ if $v_i < x_i$ and zero otherwise, we can write

$$W_{x_i} = -f_i(x_i) \left(x_i H_i(x_i) - \sum_{k=1}^{i-1} \left\{ \int_{x_k}^{\infty} v_i \prod_{\ell \neq k,i} F_\ell(\max\{v_i, x_\ell\}) dF_k(v_i) \right\} - c_i \right). \tag{14}$$

Corner solutions are not welfare maximizing as $W_{x_i} > 0$ when $v_i = 0$, and $\lim_{v_i \to \infty} W_{v_i} < 0$ due to the unboundedness of $x_i H_i(x_i)$. Therefore, an interior maximum exists, which is characterized by $W_{x_i} = 0$. It can be easily verified that the term inside the parenthesis in equation (14) is equal to zero whenever condition (1) holds. Therefore, we conclude that there exists a cutoff equilibrium that is *ex-ante* efficient. Moreover, since every equilibrium satisfies $W_{x_i} = 0$, they are either a local maximum, minimum, or a saddle point of W.

Proof of Proposition 4

Existence. If $s_1 = s_2$ the result is straight forward. Assume $s_1 < s_2$, let $g(v)$ the function implicitly defined by

$$g(v) F_1(g(v))^{m-1} F_2(v)^{n-m} = c_1.$$

The function $g(v)$ is strictly decreasing in v and satisfies $g(s_1) = s_1$. Define the function $h : [s_1, \infty) \to \mathbb{R}$ by

$$h(v) = F_2(v)^{n-m-1} \left[v F_1(v)^m - \int_{g(v)}^{v} x d(F_1(x)^m) \right] - c_2.$$

The function $h(v)$ is continuous and corresponds to the utility of a member of class 2 playing the cutoff strategy $v_2 = v$ when all other members of class 2 play v, and all members of class 1 best respond to v (i.e., $v_1 = g(v)$). The next two claims prove the result.

Claim 5. $v_2 \in [s_1, \infty)$ is necessary and sufficient to have herculean cutoffs.

Proof. $g(v)$ is weakly decreasing in v and $g(s_1) = s_1$. Therefore, $v_1 < v_2$ iff $v \geq g(v)$ iff $v_2 \in [s_1, \infty)$. □

Claim 6. $h(s_1) < 0$ and $h(v)$ is unbounded above.

Proof. Class 2 being the weak class implies $h(s_1) = s_1 F_1(s_1)^m F_2(s_1)^{n-m-1} - c_2 < 0$. On the other hand, $h(v)$ is unbounded above as $v F_1(v)^{m-1} F_2(v)^{n-m}$ is unbounded and the finite expectation assumption. □

By the intermediate value theorem, Claim 6 plus continuity imply that there exists $v^* \in (s_1, \infty)$ such that $h(v^*) = 0$. On the other hand, $h(v^*) = 0$ holds iff (6) and (11) are satisfied. Therefore, by Lemma 1 and Claim 5, we have an herculean equilibrium with $v_1 = g(v^*)$ and $v_2 = v^*$. ■

Uniqueness. The uniqueness proof is divided in three steps. Concavity is used in each of them.

Claim 7. There exists a unique herculean equilibrium.

Proof. In order to have a unique equilibrium that is herculean, it is sufficient to show that $h'(v) > 0$ for all $v \geq s_1$, so that $h(v)$ single crosses zero

$$h'(v) = F_2(v)^{n-m-1} \left\{ (n-m-1)\frac{f_2(v)}{F_2(v)} \left[vF_1(v)^m - \int_{g(v)}^{v} x dF_1(x)^m \right] \right. \\ \left. + F_1(v)^m + mg'(v)g(v)f_1(g(v))F_1(g(v))^{m-1} \right\}.$$

Since $F_2(v)^{n-m-1} > 0$ for all $v \geq s_1$, it is sufficient to show that the term in braces is positive for $v \geq s_1$. Implicitly differentiating $g(v)$

$$g'(v) = -\frac{(n-m)g(v)F_1(g(v))}{F_1(g(v)) + (m-1)g(v)f_1(g(v)} \frac{f_2(v)}{F_2(v)}$$

replacing back in to the expression in braces delivers

$$(n-m-1)\frac{f_2(v)}{F_2(v)} \left[vF_1(v)^m - \int_{g(v)}^{v} x dF_1(x)^m \right] \\ + \left[F_1(v)^m - \frac{m(n-m)g(v)^2 f_1(g(v))F_1(g(v))^m}{F_1(g(v)) + (m-1)g(v)f_1(g(v))} \frac{f_2(v)}{F_2(v)} \right]. \quad (15)$$

It is shown that a lower bound for the expression above is always positive. Maximize the subtracting term in the first square brackets by taking the upper bound $v \int_{g(v)}^{v} dF_1(x)^m$ of the integral. Using concavity ($xf(x) \leq F(x)$), maximize the subtracting term in the second square brackets by substituting $g(v)f_1(g(v))$ for $F_1(g(v))$ in the denominator. Then, equation (15) becomes

$$F_1(v)^m + [(n-m-1)v - (n-m)g(v)] F_1(g(v))^m \frac{f_2(v)}{F_2(v)} \geq F_1(v)^m \left(1 - \frac{g(v)}{v}\right)$$

where $v \geq g(v)$ for $v \geq s_1$, and $f_2(v)/F_2(v) \leq v^{-1}$ (concavity) were used to obtain the inequality. Hence the lower bound of (15) is positive iff $v \geq g(v)$, which is true as $v \geq s_1$. □

Claim 8. Symmetric players must play the same cutoffs.

Proof. By contradiction. Suppose there exists an equilibrium such that players $q < p$ are symmetric, i.e., $F_q = F_p = G$ and $c_q = c_p = c$, but the play $v_q^* < v_p^*$.

Subtracting the equilibrium condition (1) of q to the equilibrium condition of p delivers

$$0 = \sum_{j=q+1}^{p} \left\{ \left[\prod_{k \geq j, k \neq p} F_k(v_k) \right] \int_{v_{j-1}}^{v_j} \left(\prod_{\ell < q} F_\ell(v) \right) G(v) \left(\prod_{\ell = q+1}^{j-1} F_\ell(v) \right) dv \right\} - (G(v_p) - G(v_q)) \sum_{j=1}^{q} \left\{ \left[\prod_{k \geq j, k \neq q, p} F_k(v_k) \right] \int_{v_{j-1}}^{v_j} \prod_{\ell < j} F_\ell(v)\, dv \right\} \quad (16)$$

We show that a lower bound for this expression is always strictly positive, a contradiction. The first summation is strictly positive, take a lower bound by: (i) Inside the integral, for ℓ between $q+1$ and $j-1$, substitute $F_\ell(v_\ell)$ for $F_\ell(v)$; and (ii) inside the integral, substitute $F_\ell(v_q)$ and $G(v_q)$ for all other terms. Hence, the following lower bound is obtained[5]

$$\sum_{j=q+1}^{p} \left\{ \left[\prod_{k \geq j, k \neq p} F_k(v_k) \right] \int_{v_{j-1}}^{v_j} \left(\prod_{\ell < q} F_\ell(v) \right) G(v) \left(\prod_{\ell = q+1}^{j-1} F_\ell(v) \right) dv \right\} > \left(\prod_{\ell < q} F_\ell(v_q) \right) G(v_q) \left(\prod_{k > q, k \neq p} F_k(v_k) \right) (v_p - v_q)$$

It is possible to factor out the positive term $\prod_{k>q,k\neq p} F_k(v_k)$ from equation (16), reducing the expression to bound to

$$(v_p - v_q) G(v_q) \prod_{\ell < q} F_\ell(v_q) - (G(v_p) - G(v_q)) \sum_{j=1}^{q} \left\{ \prod_{k=j}^{q-1} F_k(v_k) \int_{v_{j-1}}^{v_j} \prod_{\ell < j} F_\ell(v) dv \right\} < 0$$

Now we construct an upper bound to the subtracting term. In the integral, substitute $F_\ell(v_j)$ for $F_\ell(v)$. Then, the summation in the expression above can be written as

$$\sum_{j=1}^{q-1} \left\{ v_j \left[\prod_{k=j+1}^{q-1} F_k(v_k) \right] \left(\prod_{\ell \leq j} F_\ell(v_j) - \prod_{\ell \leq j} F_\ell(v_{j+1}) \right) \right\} + v_q \prod_{\ell < q} F_\ell(v_q)$$

Since $v_j \leq v_{j+1}$, the summation in the previous expression is over non-positive terms. To obtain an upper bound replace the summation with zero and obtain the following condition

$$[v_p G(v_q) - G(v_p) v_q] \prod_{\ell < q} F_\ell(v_q) < 0$$

which holds iff $G(v_q)/v_q < G(v_p)/v_p$. A contradiction to concavity.

□

Claim 9. There is no equilibrium in which strong players play a higher cutoff than

[5]The strict inequality is guaranteed by taking $G(v_q)$ as lower bound.

weak players.

Proof. To prove that the only equilibrium is the herculean, suppose we have a non-herculean equilibrium. By Claim 8, the only possibility is to have $v_1 > v_2$ but $s_1 < s_2$. Define $\bar{g}(v)$ to be the function that satisfies

$$\bar{g}(v)F_2(\bar{g}(v))^{n-m-1}F_1(v)^m = c_2.$$

Similarly, define

$$\bar{h}(v) = F_1(v)^{m-1}\left[vF_2(v)^{n-m} - \int_{\bar{g}(v)}^{v} xd(F_2(x)^{n-m})\right] - c_1.$$

which corresponds to the utility of a player belonging to class 1, when all the members of her class play the cutoff v, and players in class 2 best respond to it. Following analogous steps to those in Claim 5 it can be shown that in order to have a non-herculean equilibrium, $\bar{h}$ has to be defined on $[s_2, \infty)$, i.e., any $v < s_2$ would deliver an herculean cutoff as a candidate and this is ruled out by assumption. Now observe that $s_1 < s_2$ implies that $\bar{h}(s_2) = s_2F_1(s_2)^{m-1}F_2(s_2)^{m-m} - c_1 > 0$. By the same argument given in Claim 7, concavity implies $\bar{h}'(v) > 0$, and $\bar{h}(v) > 0$ for all $v \in (s_2, \infty)$, so there is no v^* such that $\bar{h}\,(v^*) = 0$ and by Lemma 1 no non-herculean equilibrium exists. □

Proof of Proposition 5 *Preliminaries*: Both proofs, existence and uniqueness, use induction. In each step i, we construct the cutoff x_i as a function of an arbitrary vector $\{x_k\}_{k>i}$, treating the cutoffs of the players stronger than i—i.e., $\{x_k\}_{k<i}$—as (implicit) functions of x_i. Define $h_i^n = A_i^n D_i - c_i$. In equilibrium $h_j^n = 0$ for all j. The total differential of h_j^n with respect x_i when $j \leq j$ is given by:

$$\frac{d^T h_j^n}{dx_i} = A_j^n\left(\sum_{k=1}^{j-1} A_k^{j-1} D_k f_k(x_k)\frac{dx_k}{dx_i} + B_j(x_j)\frac{dx_j}{dx_i} + D_j\sum_{k=j+1}^{i}\frac{f_k(x_k)}{F_k(x_k)}\frac{dx_k}{dx_i}\right). \quad (17)$$

By the implicit function theorem, the vector of derivatives $X_{i-1} = (\frac{dx_1}{dx_i}, \frac{dx_2}{dx_i}, \ldots, \frac{dx_{i-1}}{dx_i})^T$, where T denotes transpose, is given by the solution to the system of linear equations $M_{i-1}X_{i-1} + d_{i-1}\frac{f_i(x_i)}{F_i(x_i)} = 0$, or $X_{i-1} = -M_{i-1}^{-1}d_{i-1}\frac{f_i(x_i)}{F_i(x_i)}$ (provided that M_{i-1} is invertible), where $d_i = (D_1, D_2, \ldots, D_i)$ and

$$M_{i-1} = \begin{pmatrix} B_1(x_1) & D_1\frac{f_2(x_2)}{F_2(x_2)} & D_1\frac{f_3(x_3)}{F_3(x_3)} & \cdots & D_1\frac{f_{i-1}(x_{i-1})}{F_{i-1}(x_{i-1})} \\ A_1^1 D_1 f_1(x_1) & B_2(x_2) & D_2\frac{f_3(x_3)}{F_3(x_3)} & \cdots & D_2\frac{f_{i-1}(x_{i-1})}{F_{i-1}(x_{i-1})} \\ A_1^2 D_1 f_1(x_1) & A_2^2 D_2 f_2(x_2) & B_3(x_3) & \cdots & D_3\frac{f_{i-1}(x_{i-1})}{F_{i-1}(x_{i-1})} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1^{i-2} D_1 f_1(x_1) & A_2^{i-2} D_2 f_2(x_2) & A_3^{i-2} D_3 f_3(x_2) & \cdots & B_{i-1}(x_{i-1}) \end{pmatrix}.$$

Observe that the common factor A_j^n in (17) can be dropped from row j in M_{i-1} and d_{i-1}. This implies that the solution of X_i is independent of n, making the induction argument feasible.

Herculean order. We show that condition (8) implies $s_i < s_{i+1}$ for any i. By definition of i's strength $s_i \prod_{j \neq i} F_j(s_i) = c_i$. Equation (8) implies $c_i < c_{i+1}F_{i+1}(s_i)/F_i(s_i)$. Substituting for c_i on the RHS of i's strength and rearranging: $s_i \prod_{j \neq i+1} F_j(s_i) < c_{i+1}$. Since the LHS is increasing in s, $s_{i+1} > s_i$. □

Existence. By induction. Order players by strength, with 1 being the strongest and n the weakest. Player one's best response as a function of the cutoffs of the other players is $x_1 = c_1/A_1^n$. Player two's profit of playing cutoff v given player one's best response is:

$$h_2^n(v) = A_2^n D_2 - c_2 = \prod_{k>2}^{n} F_k(x_k) \left[vF_1(v) - \int_{c_1/F_2(v)A_2^n}^{v} x dF_1(v) \right] - c_2.$$

The finite expectation assumption implies that $h_2^n(v)$ is unbounded above. Pick the value of v that satisfies $v = x_1$. Then, using $x_1 = c_1/A_1^n$, $h_2^n(x_1) = c_1F_1(x_1)/F_2(x_1) - c_2$. Therefore, by (8), $h_2^n(x_1) \leq 0$ (strict if $s_1 < s_2$) and, by the intermediate value theorem, there exists $x_2 \geq x_1$ (strict if $s_1 < s_2$) such that $h_2^n(x_2) = 0$. Observe that this condition holds independently of the vector $\{x_k\}_{k>2}$, implying that the order between x_1 and x_2 cannot be reversed when constructing higher cutoffs (though, the actual values of x_1 and x_2 do change).

Suppose we have shown for any profile of cutoffs $\{x_k\}_{k \geq i}$ that $x_1 \leq x_2 \leq \cdots \leq x_{i-1}$ (strict whenever $s_{j-1} < s_j$). We show that $x_{i-1} \leq x_i$ (strict if $s_{i-1} < s_i$). Recall that for any player j its profit is given by $h_j^n = A_j^n D_j - c_j$. Since player $i-1$ is best responding to i it must be the case that $h_{i-1}^n = 0$ or $D_{i-1} = c_{i-1}/A_{i-1}^n$. Let $h_i^n(v)$ be i's profit as a function of the cutoff v when all players $j < i$ are best responding to v. Because the finite expectation assumption, $h_i^n(v)$ is unbounded above. Take v to be equal to x_{i-1}. Lemma 3.2 implies $D_i = F_{i-1}(x_{i-1})D_{i-1}$, but $D_{i-1}(x_{i-1}) = c_{i-1}/A_{i-1}^n$. Then, $h_i^n(x_{i-1}) = c_{i-1}F_{i-1}(x_{i-1})/F_i(x_{i-1}) - c_i$ which is non positive under (8) and the result follows by the intermediate value theorem. Once again, the proof is independent of above cutoffs so that $x_{i-1} \leq x_i$ (strict of $s_{i-1} < s_i$) regardless of the construction of higher cutoffs. □

Uniqueness. By induction. Define $r_i = (A_1^i D_1 f_1(x_1), A_2^i D_2 f_2(x_1), \ldots, A_i^i D_i f_i(x_i))$. Fix any positive vector $\{x_k\}_{k>i+1}$ and let $\{x_k\}_{k<i+1}$ be the unique best response to x_{i+1}. We show that player $i+1$ has a unique best response to $\{x_k\}_{k>i+1}$—i.e., there is a unique value of x_{i+1} that solves $h_{i+1}^n(x_{i+1}) = 0$. In particular, we show $dh_{i+1}^n(v)/dv > 0$, so that $h_{i+1}^n(v)$ single crosses zero from below. Using (17), $dh_{i+1}^n(v)/dv = A_{i+1}^n(r_i X_i + B_{i+1}(x_{i+1}))$. Using implicit differentiation $X_i = -M_i^{-1} d_i \frac{f_{i+1}(x_{i+1})}{F_{i+1}(x_{i+1})}$. Then, $dh_{i+1}^n(v)/dv > 0$ is equivalently to show $B_{i+1}(x_{i+1}) - q_i \frac{f_{i+1}(x_{i+1})}{F_{i+1}(x_{i+1})} > 0$ where $q_i = r_i M_i^{-1} d_i$.

Start with player one. $h_1^n(v) = A_1^n v$, thus $dh_1^n(v)/dv > 0$ and has a unique best response (given by $x_1 = c_1/A_1^n$). Also, $M_1 = B_1(x_1) = 1$ is invertible. Suppose we have shown that M_{j-1} is invertible and $B_j(x_j) - q_{j-1}\frac{f_j(x_j)}{F_j(x_j)} > 0$ for all $j \leq i$—i.e., j has a unique best response. Let $p_i = \left(B_i(x_i) - q_{i-1}\frac{f_i(x_i)}{F_i(x_i)}\right)^{-1}$ and observe that $p_i > 0$ by induction hypothesis; then, by properties of partitioned matrices,

$$M_i = \begin{pmatrix} M_{i-1} & d_{i-1}\frac{f_i(x_i)}{F_i(x_i)} \\ r_{i-1} & B_i(x_i) \end{pmatrix} \text{ and } M_i^{-1} = \begin{pmatrix} O & -M_{i-1}^{-1}d_{i-1}\frac{f_i(x_i)}{F_i(x_i)}p_{i-1} \\ -r_{i-1}M_{i-1}^{-1}p_{i-1} & p_{i-1} \end{pmatrix}$$

where $O = M_{i-1}^{-1} + M_{i-1}^{-1}d_{i-1}r_{i-1}M_{i-1}^{-1}\frac{f_i(x_i)}{F_i(x_i)}p_{i-1}$, and the inverse of M_i is well defined. We need to show $B_{i+1}(x_{i+1}) - q_i\frac{f_{i+1}(x_{i+1})}{F_{i+1}(x_{i+1})} > 0$. Observing that $r_i = (r_{i-1}F_i(x_i), D_i f_i(x_i))$ and $d_i = (d_{i-1}, D_i)^T$ we can write:

$$q_i = F_i(x_i)q_{i-1} + f_i(x_i)p_i\left(D_i - q_{i-1}\right)^2 \tag{18}$$

Thus, $B_{i+1}(x_{i+1}) - q_i\frac{f_{i+1}(x_{i+1})}{F_{i+1}(x_{i+1})} > 0$ is equivalent to show:

$$\left(B_i(x_{i+1})\frac{F_i(x_{i+1})F_{i+1}(x_{i+1})}{f_i(x_i)f_{i+1}(x_{i+1})} - q_{i-1}\frac{F_i(x_i)}{f_i(x_i)}\right)(B_i(x_i) - q_{i-1}\frac{f_i(x_i)}{F_i(x_i)}) > \left(D_i - q_{i-1}\right)^2$$

where $B_{i+1}(x_{i+1}) = B_i(x_{i+1})F_i(x_{i+1})$ and the definition of p_i were used. Using $x_i \leq x_{i+1}$ and concavity ($F(x)/f(x) > x$), we can find a lower bound for the LHS of the expression above: $(B_i(x_i)x_i - q_{i-1})^2$. Lemma 3.1 shows $B_i(x_i)x_i \geq D_i$. Thus we just need to show that $B_i(x_i)x_i - q_{i-1} \geq 0$, which is done by proving $D_i - q_{i-1} \geq 0$. We do this by induction. Since q_0 is not defined, we start with $i = 2$. Using integration by parts $D_2 - q_1$ is equal to

$$x_1F_1(x_1) + \int_{x_1}^{x_2} F_1(s)ds - (x_1)^2 f_1(x) > \int_{x_1}^{x_2} F_1(s)ds \geq 0$$

were concavity was used in the last step. Suppose we have shown $D_j \geq q_{j-1}$ for $j \leq i$. We need to show $D_{i+1} \geq q_i$. Using equation (18), this is equivalent to:

$$\frac{D_{i+1}}{F_i(x_i)} - q_{i-1} - \frac{f_i(x_i)}{F_i(x_i)}p_i\left(D_i - q_{i-1}\right)^2 \geq 0.$$

Lemma 3.2 shows $D_{i+1}/F_i(x_i) \geq D_i$. By induction hypothesis $D_i \geq q_{i-1}$, using the definition of p_i we can rewrite the condition as

$$1 \geq \frac{D_i - q_{i-1}}{B_i(x_i)\frac{F_i(x_i)}{f_i(x_i)} - q_{i-1}}.$$

The result follows from concavity and the fact that $x_iB_i(x_i) \geq D_i$. Thus $D_{i+1} \geq q_i$, which proves $dh_{i+1}^n(v)/dv > 0$ and a unique herculean equilibrium exists. □

Proof of Proposition 6 *Preliminaries*: If $s_1 = s_2$ the herculean equilibrium corresponds to the strength of the firms. Assume $s_1 < s_2$ and define:

$$H_i(v_i, v_j) = \pi_i(v_i)F_j(v_j) + \int_{v_j}^{\infty} \pi_i(v_i, x)dF_j(x) - c_i.$$

Let $g(x)$ be the function that solves $H_1(g(x), x) = 0$. $g(x)$ is well defined by intermediate value theorem: A3 implies that $H_1(0, x) < 0$ and that exits v such that $H_1(v, x) > 0$. Thus, since $H_i(v_i, v_j)$ is continuous and strictly increasing in v_i, for each x there exists a unique value $g(x)$ satisfying $H_1(g(x), x) = 0$.

Claim 10. $g(s_1) = s_1$. $g'(x) < 0$ and, under (10), $g'(x)$ is bounded below by

$$-\frac{f_2(x)F_1(g(x))}{F_2(x)f_1(g(x))}. \tag{19}$$

Proof. When $x = s_1$, $H_1(g(s_1), s_1)$ coincides with the definition of strength, implying $g(s_1) = s_1$. Using the implicit function theorem:

$$g'(x) = -\frac{f_2(x)\Delta_1(g(x), x)}{F_2(x)\pi_1'(g(x)) + \int_v^{\infty} \pi_1'(g(x), y)dF_2(y)}$$

which is negative. For the bound observe that the integral in the denominator is positive, and that condition (10) implies $\Delta_1(g(x), x) < F_1(g(x))\pi_1'(g(x))/f_1(g(x))$, then

$$g'(x) > -\frac{f_2(x)\Delta_1(g(x), x)}{F_2(x)\pi_1'(g(x))} > -\frac{f_2(x)F_1(g(x))}{F_2(x)f_1(g(x))}.$$

□

Proof of Existence: Define the function $h : [s_1, \infty) \to \mathbb{R}$ by $h(x) = H_2(x, g(x))$. This function is continuous and corresponds to the expected profits of firm two participating in the market and firm one plays the best response cutoff. Define x_2 to be the value $\hat{x}$ satisfying $h(\hat{x}) = 0$, we prove that $\hat{x}$ exists and that is an herculean equilibrium. The next two claims prove the result.

Claim 11. $\hat{x} \in [s_1, \infty)$ is necessary and sufficient to have herculean cutoffs

Proof. Since $x_1 = g(x)$ is decreasing in x and $g(s_1) = s_1$ we have: $x_1 < x_2 \Leftrightarrow g(x) < x \Leftrightarrow x_2 \in [s_1, \infty)$. □

Claim 12. $h(s_1) < 0$ and $\lim_{x\to\infty} h(x) > c_2$. Thus, $h(x) = 0$ exists.

Proof. Because firm two is weak $h(s_1) = H_2(s_1, s_1) < H_2(s_2, s_2) = 0$. For the limit, observe

$$\begin{aligned} h(x) &= F_1(g(x))\pi_2(x) + \int_{g(x)}^{\infty} \pi_2(x, y)dF_j(y) - c_2 \\ &> \int_0^{\infty} \pi_2(x, y)dF_j(y) - c_2 \end{aligned}$$

and the result follows from A3. □

Proof for Uniqueness: We start by proving uniqueness within the herculean class of equilibrium. It is shown that $h'(x) > 0$ so that $h(x)$ single crosses zero from below. The derivative of $h(x)$ is:

$$h'(x) = \pi_2'(x)F_1(g(x)) + \int_{g(x)}^{\infty} \pi_2'(x,y)\, dF_j(y) + g'(x)f_1(g(x))\,\Delta_2(x,g(x)).$$

The first two terms of $h'(x)$ are positive, and only the term containing $g'(x)$ is negative. Replacing the lower bound (19) we find

$$h'(x) > \pi_2'(x)F_1(g(x)) + \int_{g(x)}^{\infty} \pi_2'(x,y)dF_1(y) - \frac{f_2(x)\Delta_2(x,g(x))F_1(g(x))}{F_2(x)}$$

Condition (10) implies $f_2(x)\Delta_2(x,g(x)) < F_2(x)\pi_2'(x)$, then:

$$h'(x) > \int_{g(x)}^{\infty} \pi_2'(x,y)dF_1(y) > 0$$

which proves uniqueness within the herculean class. To show that there is no non-herculean equilibrium [fill this up]

Proof of Proposition 7 It follows from observing that, for all x and y:

$$\frac{f_i(x)}{F_i(x)}\frac{\Delta_i(x,y)}{\pi_i'(x)} < \frac{f_i(x)}{F_i(x)}\frac{\pi_i(x)}{\pi_i'(x)} < 1$$
$$\frac{f_i(x)}{F_i(x)}\frac{\Delta_i(x,y)}{\pi_i'(x)} < \frac{f_i(x)}{F_i(x)}\frac{\Delta_i(x,y)}{\Delta_i'(x,y)} < 1$$

Proof of Proposition 8 In general terms, the ex-ante expected profits of players with cutoffs $x_1 \leq x_2$ can be written as

$$\mathbb{E}[\Pi_1] = \int_{x_1}^{\infty}\left(\pi_1(x)F_2(x_2) - \int_{x_2}^{\infty}\pi_1(x,y)dF_2(y) - c_1\right)dF_1(x)$$
$$\mathbb{E}[\Pi_2] = \int_{x_2}^{\infty}\left(\pi_2(x)F_1(x_1) - \int_{x_1}^{\infty}\pi_2(x,y)dF_1(y) - c_2\right)dF_2(x)$$

Under symmetry we have that $F_1(x) = F_2(x) = F$, $c_1 = c_2 = c$, $\pi_1(x) = \pi_2(x)$ and $\pi_1(x,y) = \pi_2(x,y)$. Subtracting $\mathbb{E}[\Pi_2]$ to $\mathbb{E}[\Pi_1]$ we obtain:

$$\mathbb{E}[\Pi_1] - \mathbb{E}[\Pi_2] = \int_{x_1}^{x_2}\left(\pi(x)F(x_2) - \int_{x_2}^{\infty}\pi(x,y)dF(y) - c\right)dF(x) +$$
$$\int_{x_2}^{\infty}\left(\int_{x_1}^{x_2}(\pi(x) - \pi(x,y))\,dF(y)\right)dF(x).$$

The equilibrium condition for player 1 implies that the term inside the first integral is positive for all $x > x_1$ and zero for $x = x_1$. Hence, the first term is positive. Since $\pi(x, y) \leq \pi(x)$, the term inside the second integral is non-negative. Thus, $\mathbb{E}[\Pi_1] > \mathbb{E}[\Pi_2]$

Proof of Proposition 9 *Preliminaries*: If $s_1 = s_2$ the herculean equilibrium corresponds to the strength of the firms. Assume $s_1 < s_2$ and define $H_i(x_i, x_j)$ to be equal to:

$$\sum_{k=0}^{m_j} \left\{ \binom{m_j}{k} F_j(x_j)^{m_j-k} \left[\sum_{r=0}^{m_i-1} \binom{m_i-1}{r} F_i(x_i)^{m_i-1-r} \mathbb{E}[\pi_i(x_i)|\mathbf{x}, r, k] \right] \right\} - c_i. \quad (20)$$

Let $g(x)$ be the function that solves $H_1(g(x), x) = 0$. As before, $g(x)$ is well defined by intermediate value theorem.

Claim 13. $g(s_1) = s_1$. $g'(x) < 0$ and, under (10), $g'(x)$ is bounded below by (19).

Proof. Once again, the definition of strength implies $g(s_1) = s_1$. Using the implicit function theorem

$$g'(x) = -\frac{\partial H_1(g(x), x)/\partial x_2}{\partial H_1(g(x), x)/\partial x_1},$$

which is negative by Lemma X. For the lower bound observe that the Δ_1 terms in $\partial H_1(x_1, x_2)/\partial x_1$ are positive so that

$$\begin{aligned} g'(x) &> \frac{-f_2(x) \sum_{k=0}^{m_2} \left\{ \binom{m_2}{k} (m_2 - k) F_2(x)^{m_2-k-1} \mathbb{E}_r \left[\mathbb{E}[\Delta_1(g(x), x)|\mathbf{x}, r, k] \right] \right\}}{\sum_{k=0}^{m_2} \left\{ \binom{m_2}{k} F_2(x)^{m_2-k} \left[\sum_{r=0}^{m_1-1} \binom{m_1-1}{r} F_1(g(x))^{m_1-1-r} \mathbb{E}[\pi_1'(g(x))|\mathbf{x}, r, k] \right] \right\}} \\ &= \frac{-f_2(x) \sum_{k=0}^{m_2} \left\{ \binom{m_2}{k} (m_2 - k) F_2(x)^{m_2-k} \mathbb{E}_r \left[\mathbb{E}[\Delta_1(g(x), x)|\mathbf{x}, r, k] \right] \right\}}{F_2(x) \sum_{k=0}^{m_2} \left\{ \binom{m_2}{k} F_2(x)^{m_2-k} \mathbb{E}_r [\mathbb{E}[\pi_1'(g(x))|\mathbf{x}, r, k]] \right\}} \end{aligned}$$

Using Lemma X,

$$\mathbb{E}[\Delta_1(g(x), x)|\mathbf{x}, r, k] < F_1(g(x)) \mathbb{E}[\pi_1'(g(x))|\mathbf{x}, r, k]/(m_2 - k) f_1(g(x))$$

substituting and canceling out terms (19) is obtained. □

Define the function $h : [s_1, \infty) \to \mathbb{R}$ by $h(x) = H_2(x, g(x))$. This function is continuous and corresponds to the expected profits of firm two participating in the market and firm one plays the best response cutoff. Define x_2 to be the value of x such that $h(x_2) = 0$, we prove that x_2 exists and that is an herculean equilibrium. The next claims prove the results.

Claim 14. $v_2 \in [s_1, \infty)$ is necessary and sufficient to have herculean cutoffs

Proof. Since $g(x)$ is decreasing in x and $g(s_1) = s_1$ we have, $x_1 < x_2 \iff x > g(x) \iff x_2 \in [s_1, \infty)$. □

Claim 15. $h(s_1) < 0$ and $\lim_{v\to\infty} h(v) > c_2$.

Proof. Because player two is weak $h(s_1) = H_2(s_1, s_1) < H_2(s_2, s_2) = 0$. The limit follows from $A3$ and the observation:

$$\begin{aligned} h(v) &= F_1(g(v))\,\pi_2(v) + \int_{g(v)}^{\infty} \pi_2(v,x)\,dF_j(x) - c_2 \\ &> \mathbb{E}_x[\pi_2(v,x)] - c_2 \end{aligned}$$

□

Uniqueness. To type!

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